ESTIMATING THE EXISTENCE DOMAIN OF A LIMITING CYCLE OF A CLASS OF DYNAMIC SYSTEMS WITH A CYLINDRICAL PHASE SPACE

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Several problems in the theory of oscillations necessitate investigation of nonlinear second-order differential Eqs. of the form

$$\varphi'' + F(\varphi)\varphi' + f(\varphi) = T \tag{1}$$

where $F(\varphi)$ and $f(\varphi)$ are certain periodic functions. The phase space of the system

$$\frac{d\varphi}{dt} = y, \qquad \frac{dy}{dt} = T - f(\varphi) - F(\varphi)\, y \tag{2}$$

corresponding to Eq. (1) is a cylinder. In investigating the qualitative structure of the decomposition of the phase cylinder into trajectories it is often important to isolate in the space of system parameters the existence domain of the limiting cycle which includes the phase cylinder. Usually this is effected by investigating the equilibrium states of system (2) and studying the behavior of the saddle separatrices. As a rule this is an intricate and time-consuming operation.

We propose to show how in certain cases one can make use of almost self-evident considerations involving a properly chosen "comparison system" generated by Eq.

$$\varphi'' + \Phi(\varphi)\varphi'^2 + f(\varphi) = T \tag{3}$$

to obtain a sufficient criterion for the existence of a limiting cycle encompassing the phase cylinder and to estimate the existence domain in the parameter space without investigating the behavior of the saddle separatrices.

Eq. (3) is associated with the system

$$\frac{d\varphi}{d\tau} = y, \qquad \frac{dy}{d\tau} = T - f(\varphi) - \Phi(\varphi)\, y^2 \tag{4}$$

Let us consider the phase half-cylinder $y > 0$ of systems (2) and (4). It does not contain equilibrium states, since these all lie along the y-axis.

If infinity is unstable for system (2), if in some domain $G\,(0 < y_1 < y < \infty)$ of the cylindrical phase space the vector field corresponding to system (2) is rotated at each point by a positive angle relative to the vector field corresponding to system (4), and if system (4) has a stable limiting cycle in the domain G, then system (2) also has a stable limiting cycle in G. This statement is almost self-evident. The trajectory $L(t - t_0)$ of system (2) which for $t = t_0$ and $\tau = \tau_0$ intersects at a positive angle the trajectory $L^*(\tau - \tau_0)$ of system (4) coiled from above onto the limiting cycle in the domain G cannot intersect the limiting cycle for $t = t_1 > t_0$ and $\tau = \tau_1 > \tau_0$, since this would contradict the sign of rotation of the vector field and thus could not proceed to the equilibrium state. Since the trajectory cannot go to infinity (infinity being unstable), it must coil onto the limiting cycle lying above the limiting cycle of system (4).

The effective use of system (4) as a comparison system rests on the possibility of its direct integration and the consequent possibility of finding the equation of the limiting cycle of the comparison system if such a cycle exists.

Example 1. Let us consider system [1 to 3]

$$\frac{d\varphi}{dt}=y, \qquad \frac{dy}{dt}=T-\sin\varphi-r\sin 2\varphi-h(1-b\cos 2\varphi)\,y, \qquad |b|<1 \tag{5}$$

and the comparison system

$$\frac{d\varphi}{dt}=y, \qquad \frac{dy}{dt}=T-\sin\varphi-r\sin 2\varphi-hy^2 \tag{6}$$

Here T, r, h are positive parameters. System (6) has the following family of integral curves on the cylindrical phase space:

$$y^2=\frac{T}{h}-\frac{2}{\sqrt{1+4h^2}}\sin(\varphi+\theta_0)-\frac{r}{\sqrt{1+h^2}}\sin(2\varphi+\theta_1)+ce^{-2h\varphi} \tag{7}$$

The value $c = 0$ in (7) is associated with a stable limiting cycle of system (6) if the values of the system parameters are such that the right-hand side of (7) for $c = 0$ remains positive for all φ. The trajectories of systems (5) and (6) are tangent along the "contact curve" $1 - b\cos 2\varphi = y$, whose largest ordinate is $1+|b|$. The vector field of system (5) is rotated in the positive direction relative to the vector field of system (6) for all $y > 1 + |b|$. The following estimate is clearly valid for the minimum ordinate of the limiting cycle of system (6):

$$y_{\min}^2>\frac{T}{h}-\frac{2}{\sqrt{1+4h^2}}-\frac{r}{\sqrt{1+h^2}}$$

The inequality

$$\frac{T}{h}-\frac{2}{\sqrt{1+4h^2}}-\frac{r}{\sqrt{1+h^2}}\geqslant(1+|b|)^2 \tag{8}$$

isolates in the parameter space $T > 0$, $r > 0$, $h > 0$, $|b| < 1$ a domain for whose points system (5) is known in advance to have a limiting cycle (since $h > 0$ and $|b| < 1$, infinity is unstable for system (5)).

Example 2. Let us consider a system describing the auto-oscillations of a synchronous motor,

$$\frac{d\varphi}{dt}=y, \qquad \frac{dy}{dt}=T-\sin\varphi-(A+B\sin^2\varphi-\gamma\sin\varphi)\,y \tag{9}$$

Here T and A are positive parameters. Eq. (9) is investigated in [4 to 6] by the small parameter method. Making use of the comparison system

$$\frac{d\varphi}{dt}=y, \qquad \frac{dy}{dt}=T-\sin\varphi-\lambda y^2 \tag{10}$$

with a still undetermined positive parameter λ, we can, without introducing a small parameter into (9), isolate in the parameter space $T > 0$, $A > 0$, B and γ some domain for whose points system (9) has a stable limiting cycle. Let us limit ourselves to the case $T < 1$, which is of the greatest interest. System (10) has the following family of integral curves on the cylindrical phase space:

$$y^2=ce^{-2\lambda\varphi}+\frac{T}{\lambda}+\frac{2}{\sqrt{4\lambda^2+1}}\sin(\varphi+\varphi_0) \tag{11}$$

The value $c = 0$ in (11) is associated with a stable limiting cycle if

$$V_0=y_{\min}^2\equiv\frac{T}{\lambda}-\frac{2}{\sqrt{4\lambda^2+1}}>0$$

The contact curve for systems (9) and (10) is

$$y=\lambda^{-1}(A+B\sin^2\varphi-\gamma\sin\varphi) \tag{12}$$

We note that the condition of infinity instability for system (9) (the coefficient of y assumes positive values only) coincides with the condition $y > 0$ for contact curve (12). We require that the contact curve be situated on the phase cylinder in the domain $y > 0$ and "below" the limiting cycle of comparison system (10). For the values of the parameters T, A, B, and γ for which this requirement is fulfilled, i.e. in the domain "above" the limiting cycle of system (10), the vector field of system (9) is rotated positively relative to the vector field of system (10), infinity is unstable for (9), and system (9) has at least one stable limiting cycle in the domain under consideration.

In order for system (9) to have a limiting cycle it is sufficient that the inequalities $f(A, B, \gamma) > 0$ and $F^2(A, B, \gamma) < y_{\min}^2 \equiv V_0$ be fulfilled, where $f(A, B, \gamma)$ and $F(A, B, \gamma)$ are the minimum and maximum of contact curve (12), respectively. These inequalities lead to the two following groups of inequalities, respectively:

$$\begin{aligned} A + B - \gamma &> 0 \quad \text{for } \gamma - 2B \geqslant 0,\ \gamma \geqslant 0 \\ A + B + \gamma &> 0 \quad \text{for } \gamma + 2B \leqslant 0,\ \gamma \leqslant 0 \\ A - {}^1\!/_4\,\gamma^2 / B &> 0 \quad \text{for } \gamma - 2B \leqslant 0,\ \gamma + 2B \geqslant 0 \end{aligned} \tag{13}$$

$$\begin{aligned} (A + B - \gamma)^2 &< c_0^2 \quad \text{for } \gamma - 2B \leqslant 0,\ \gamma \leqslant 0 \quad (c_0 = \lambda y_{\min}) \\ (A + B + \gamma)^2 &< c_0^2 \quad \text{for } \gamma + 2B \geqslant 0,\ \gamma \geqslant 0 \\ (A - {}^1\!/_4\,\gamma^2 / B)^2 &< c_0^2 \quad \text{for } \gamma - 2B \geqslant 0,\ \gamma + 2B \leqslant 0 \end{aligned} \tag{14}$$

The domain isolated by inequalities (13) and (14) in the space of the parameters T, A, B and γ depends essentially on the parameter λ of comparison system (10). The permissible values of λ lie in the range $0 < \lambda < T/2\sqrt{1 - T^2}$ corresponding to the condition $V_0 > 0$. In order for the domain of values of the parameters A, B, γ to be maximum for some fixed $T < 1$, it is necessary to choose a λ for which the curve

$$c_0^2(\lambda) \equiv T\lambda - \frac{2\lambda^2}{\sqrt{4\lambda^2 + 1}}$$

has a maximum. This yields the condition

$$T\,(4\lambda^2 + 1)^{3/2} - 4\lambda\,(2\lambda^2 + 1) = 0$$

which enables us to determine λ unambiguously from the given $T < 1$.

In Fig. 1 we have isolated in the plane of the parameters B and γ for fixed $A = 0.18$, $T = 2/3$ a domain for whose points system (9) has a stable limiting cycle encompassing the phase cylinder. The value $T = 2/3$ corresponds to $\lambda = 0.19$ for which $c_0^2(\lambda)$ has its maximum value.

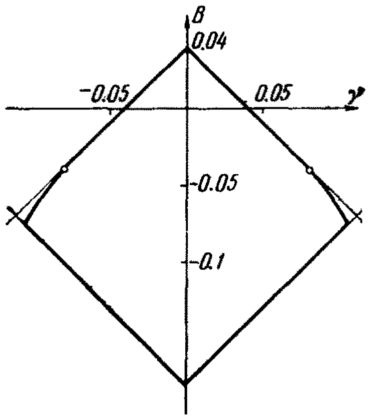

Fig. 1

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BIBLIOGRAPHY

1. Ku, Y.H., Nonlinear analysis of electromechanical problems. J. Franklin Inst. Vol. 255, 1953.
2. Beliustina, L.N., An Equation of Electrical Machine Theory, in the collection In Memoriam A.A. Andronov, Izd. Akad. Nauk SSSR, 1955.
3. Ku, Y.H., Analysis and Control of Nonlinear Systems, Ronald, New York, 1958.
4. Vlasov, N.P., Auto-oscillations of a synchronous motor. Uch. zap. Gor'kovsk. Univ. Vol. 12, No. 3, 1939.
5. Gersht, E.N., Qualitative investigation of a differential equation of electrical machine theory. Izv. Akad. Nauk SSSR. Mekhan. i mashinostroenie No. 1, 1964.
6. Gersgt, E.N., Qualitative investigation of a differenti al equation of the theory of electrical devices. Inzh. zh. MTT No. 1, 1966.

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